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Off-Shell Extended Supersymmetries and Lorentz-Violating Abelian Gauge Models

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Abstract

In this work, we propose the $N = 2$ and $N = 4$ supersymmetric extensions realized off-shell of the Abelian gauge model with Chern-Simons Lorentz-breaking term. We start with the theory in 6 and 10 dimensions and reduce à la Scherk the space-like coordinates to carry out the $D = 5$ model in both cases. Then, we reduce the fifth space-like coordinate using the Legendre transformation technique for dimensional reduction. The last reduction method provides us with auxiliary fields that yield the superalgebra closed off-shell. Since the reduced bosonic Lagrangians from $6D$ and $10D$ are the same as the $N = 2$ and $N = 4$ SUSY-versions of the theory, respectively, we use the superspace-superfield formalism in $N = 1$ to achieve the supersymmetric version of model.

1 Introduction

The field-theoretic models adopted to describe the fundamental interactions among truly elementary particles have the Lorentz and gauge symmetries as their main cornerstones. However, mechanisms of breaking these symmetries have been proposed and discussed in view of some phenomenological and experimental evidences [1, 2, 3, 4, 5]. Astrophysical observations indicate that Lorentz symmetry may be slightly violated in order to account for anisotropies. Then, one may consider a gauge theory where Lorentz symmetry breaking may

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be realized by means of a term in the action. A Chern-Simons-type term may be considered that exhibits a constant background four-vector which maintains the gauge invariance but breaks down the Lorentz space-time symmetry [1].

In the context of supersymmetry (SUSY), a number of works introduce the idea of Lorentz breaking in connection with SUSY: In ref. [6], SUSY was presented through the modification of the algebra; in ref. [7, 8], one achieves the $N = 1-$SUSY version of the Chern-Simons term through a superspace-superfield formalism; in ref. [9], the authors adopt the idea of Lorentz breaking operators. The $N = 2$ and $N = 4$ extended supersymmetric version of the Lorentz breaking term have been presented in [10] using the ordinary dimension reduction (à la Scherk) of the bosonic sector in 10 and 6 space-time dimensions respectively and then using the superspace-superfield formalism in $N = 1$. In this last one, the extended supersymmetrizations are realized on shell, i.e., the superalgebra can not close without imposition of the equations of motion in 4D. In the present paper, we propose the off-shell $N = 2$ and $N = 4$ supersymmetric version of the Abelian gauge sector added the Chern-Simons Lorentz breaking term. The main idea is based on the work of ref. [11], where it was presented the $N = 2$ and $N = 4$ supersymmetric version realized off-shell for the Yang-Mills theory. It was obtained through the dimensional reduction of the $N = 1$ supersymmetric model in components in 6 and 10 dimensions respectively. In order to obtain the off-shell realization, different of the refs. [12, 10] presented on-shell, it proceeds making the ordinary dimensional reduction of space-like coordinates from the $N = 1$ model in $6D$ and $10D$ to $5D$ and than reducing the other space-like coordinate to $4D$ using the Legendre transformation technique. The model receive some auxiliary fields, that have close relations with the conjugate momenta, that make superalgebra closed off-shell and the translation in respect to the reduced coordinate (à la Legendre) can be identified as central charge transformation.

In the ordinary dimension reduction, presented by Scherk, one assumes that no field depends on the coordinate to be reduced, then all derivatives with respect to this coordinates are taken to be zero. The technique of reduction à la Legendre is not so direct: its main idea is based on the Hamiltonian, not with respect to the time, but with respect to the coordinate to be reduced. In this technique, the fields and the Lagrangian are dependent of the extra coordinate, but the reduced action does not have this dependence. In this way, the fields have to obey the equation of motion in the higher dimension that turns into constraints of the reduced Lagrangian. In the first section, we shall present these techniques and then we shall apply to a $D = 6$ and a $D = 10$ Abelian gauge model with Lorentz-violating Chern-Simons term. In both applications, we reduce the space-like coordinates à la Scherk to achieve the $D = 5$ version of the model, and then, we reduce more one space-like coordinate using the Legendre transformation technique. In the second and the third sections, we use the fact that the reduced Lagragians starting with $D = 6$ and $D = 10$ model, obtained in the first section, are the bosonic sectors of the off-shell $N = 2$ and $N = 4$ supersymmetric version of the model proposed, respectively. Once the bosonic sector is identified, we adopt an $N = 1$ -superfield formalism to write down the gauge and the background supermultiplets and then we set up their coupling in terms of an $N = 2$ and $N = 4$ action realized in $N = 1$-superspace. The result is projected out in component fields and we discuss the role of the background

partners for the central charge of the $N = 2$ and $N = 4$ Lorentz broken action.

2 The Scherk and Legendre procedures for dimensional reduction

There are in the literature several techniques for dimensional reduction, such as à la Scherk [13, 14], à la Legendre [11], à la Kaluza-Klein [15, 16], à la Witten-Manton [17, 18, 19], and others. In this work, we shall contemplate two of them: the technique à la Scherk and à la Legendre to make the dimensional reduction of the Abelian gauge model with a Lorentz violating term. This shall be useful to achieve, in the next sections, the off-shell $N = 2$ and $N = 4$ supersymmetric version of this model. This model in $4D$ was proposed by [1], and it is written as follows:

$$\mathcal{L}_4 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa t_\lambda, \tag{2.1}$$

where t_λ is a constant vector that exist in the background space-time, determining a preference direction, and then, breaking the Lorentz symmetry.

In the first part of this section, we shall present the dimensional reduction techniques à la Scherk (ordinary technique) and à la Legendre (through Legendre transformations). Then, in the second part, we shall apply these techniques to make the dimensional reduction from $D = 6$ to $D = 4$ and from $D = 10$ to $D = 4$ for the $D = 6$ and $D = 10$ versions of (2.1). This procedure will provide the bosonic sector for the $N = 2-$ and $N = 4-$SUSY generalization of this model, respectively. Then, in the next sections, we will carry out these complete off-shell supersymmetric model using the superfields formalism in a $N = 1$ superspace.

The ordinary reduction (à la Scherk) is very direct, it is considered that the fields do not depend on the extra coordinate to be reduced ($\partial_4 A_{\hat{\mu}} = 0$). In the reductions $D = 6$ to $D = 4$ and $D = 10$ to $D = 4$, we will proceed reducing à la Scherk the space-like coordinates to $D = 5$, and then, we will reduce the space-like coordinate x^4using the Legendre transformation technique.

In the Legendre transformation technique, it is supposed that the Lagrangian $\mathcal{L}_4$ has x^4-dependence ($\partial_4\mathcal{L}_4 \neq 0$), but the action S_4 does not depend on x^4 ($\partial_4 S_4 = 0$). This kind of reduction is interesting also because we suppose that physically the fields really depend on the extra dimension.

After we obtain the Lagrangian in $5D$, we make the dimensional reduction with respect to a coordinate x^4 through the Legendre transformation technique. It is based on the idea of the Hamiltonian formalism. The Hamiltonian density is the Legendre transformation of the Lagrangian (density) with respect to the time and we know that the Hamiltonian density has time dependence ($\frac{d}{dt}\mathcal{H} \neq 0$), but the Hamiltonian is invariant under time translations ($\frac{d}{dt}H = 0$). The idea is to obtain the Lagrangian $\mathcal{L}_4$ as a kind of "Hamiltonian density" with respect to the coordinate x^4. The Lagrangian $\mathcal{L}_4$, unlike the à la Scherk method, depends

on the coordinates x^4:

$$\frac{\partial \mathcal{L}_4}{\partial x^4} \neq 0. \tag{2.2}$$

However, the action must be independent of x^4:

$$\frac{\partial S_4}{\partial x^4} = 0. \tag{2.3}$$

In order to find the Lagrangian $\mathcal{L}_4$ from a Lagrangian $\mathcal{L}_5$ we make a procedure similar to obtain the Hamiltonian, but unlike the Hamiltonian, it is not with respect to the time, but to the coordinate x^4. Suppose the Lagrangian in 5 dimensions $\mathcal{L}_5$ doesn't have explicit x^4 dependence (only implicity):

$$\frac{\partial \mathcal{L}_5}{\partial x^4} = \frac{\partial \mathcal{L}_5}{\partial A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} + \frac{\partial \mathcal{L}_5}{\partial \partial_{\hat{\mu}} A_{\hat{\nu}}} \frac{\partial \partial_{\hat{\mu}} A_{\hat{\nu}}}{\partial x^4}, \tag{2.4}$$

where $\hat{\mu} = 0, 1, 2, 3, 4$. If we make the integration by parts obtaining

$$\frac{\partial \mathcal{L}_5}{\partial x^4} = \left(\frac{\partial \mathcal{L}_5}{\partial A_{\hat{\nu}}} - \partial_{\hat{\mu}} \frac{\partial \mathcal{L}_5}{\partial \partial_{\hat{\mu}} A_{\hat{\nu}}} \right) \frac{\partial A_{\hat{\nu}}}{\partial x^4} + \partial_{\hat{\mu}} \left(\frac{\partial \mathcal{L}_5}{\partial \partial_{\hat{\mu}} A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right). \tag{2.5}$$

Considering that the field equation in $5D$ is satisfied, then

$$\frac{\partial \mathcal{L}_5}{\partial A_{\hat{\nu}}} - \partial_{\hat{\mu}} \frac{\partial \mathcal{L}_5}{\partial \partial_{\hat{\mu}} A_{\hat{\nu}}} = 0 \tag{2.6}$$

and (2.5) becomes

$$\frac{\partial \mathcal{L}_5}{\partial x^4} = \partial_{\hat{\mu}} \left(\frac{\partial \mathcal{L}_5}{\partial \partial_{\hat{\mu}} A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right). \tag{2.7}$$

Considering $\mu = 0, 1, 2, 4$, we have that

$$\frac{\partial \mathcal{L}_5}{\partial x^4} = \partial_{\mu} \left(\frac{\partial \mathcal{L}_5}{\partial \partial_{\mu} A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right) + \partial_4 \left(\frac{\partial \mathcal{L}_5}{\partial \partial_4 A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right). \tag{2.8}$$

Then

$$\frac{\partial}{\partial x^4} \left(\mathcal{L}_5 - \frac{\partial \mathcal{L}_5}{\partial \partial_4 A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right) = \partial_{\mu} \left(\frac{\partial \mathcal{L}_5}{\partial \partial_{\mu} A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right). \tag{2.9}$$

If we integrate this expression in the 4 space-time volume:

$$\frac{\partial}{\partial x^4} \int d^4x \left(\mathcal{L}_5 - \frac{\partial \mathcal{L}_5}{\partial \partial_4 A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right) = \int d^4x \partial_{\mu} \left(\frac{\partial \mathcal{L}_5}{\partial \partial_{\mu} A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} \right), \tag{2.10}$$

we notice that the term in the RHS of equation (2.10) must turn into a hipersurface integral that must be zero, because $A_{\hat{\nu}}$ must be fix. Then, we can define a Lagrangian $\mathcal{L}_4$ that can be thought of as the negative of the "Hamiltonian" with respect to the x^4 as:

$$\mathcal{L}_4 = -\mathcal{H}_5 = \mathcal{L}_5 - \frac{\partial \mathcal{L}_5}{\partial \partial_4 A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4}. \tag{2.11}$$

In this reduced Lagrangian, new auxiliary fields appear; they are closely related to the canonical momenta

$$\pi^{\hat{\nu}} = \frac{\partial \mathcal{L}_4}{\partial \partial_4 A_{\hat{\nu}}}. \tag{2.12}$$

These auxiliary fields have the role to close the off-shell superalgebra in one dimension less.

We notice that to define (2.11) as a Lagrangian in $4D$, the equations of motion in $5D$ (2.6) must be satisfied. It becomes a constraint in $5D$ that determine how the physical and auxiliary fields transform under x^4 translations. This x^4 translations correspond to the central-charge transformations in the reduced superalgebra.

2.1 The chain of reductions $6D \rightarrow 5D \rightsquigarrow 4D$ [2]

In [11], it was shown the attainment of the off-shell $N = 2-$SUSY generalization for the Yang-Mills model applying the two techniques of dimensional reduction mentioned for the $N = 1-$SUSY model in components in $D = 6$. Here, we shall apply these dimensional reduction techniques for a $D = 6$ version of the bosonic model (2.1). The reduced Lagrangian is the bosonic sector of the off-shell $N = 2-$SUSY generalization of this model. As proposed in [10], the $D = 6$ version of the Lagrangian (2.1) can be given by:

$$\mathcal{L}_6 = -\frac{1}{4} F_{\dot{\mu}\dot{\nu}} F^{\dot{\mu}\dot{\nu}} + \frac{1}{3!} \varepsilon^{\dot{\mu}\dot{\nu}\dot{\kappa}\dot{\rho}\dot{\lambda}\dot{\sigma}} A_{\dot{\mu}} \partial_{\dot{\nu}} A_{\dot{\kappa}} T_{\dot{\lambda}\dot{\rho}\dot{\sigma}}, \tag{2.13}$$

where $\dot{\mu} = 0, 1, 2, 3, 4, 5$.

The dimensional reduction of a space-like coordinate x^5 à la Scherk is done considering no dependence of the fields in respect to this coordinates. This reduction takes the form as follows:

$$\mathcal{L}_5 = -\frac{1}{4} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} + \frac{1}{2} \partial_{\hat{\mu}} \varphi \partial^{\hat{\mu}} \varphi + \frac{1}{2} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}} A_{\hat{\mu}} \partial_{\hat{\nu}} A_{\hat{\kappa}} R_{\hat{\lambda}\hat{\rho}} + \frac{1}{3!} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}} A_{\hat{\mu}} \partial_{\hat{\nu}} \varphi T_{\hat{\kappa}\hat{\lambda}\hat{\rho}}, \tag{2.14}$$

where $\hat{\mu} = 0, 1, 2, 3, 4, \quad \varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}5} \equiv \varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}$ and we redefined the gauge fields as

$$(A_{\dot{\mu}}) \rightarrow (A_{\hat{\mu}},\, \varphi),$$

and the background fields as

$$(T_{\dot{\lambda}\dot{\rho}\dot{\sigma}}) \rightarrow (R_{\hat{\lambda}\hat{\rho}},\, T_{\hat{\kappa}\hat{\lambda}\hat{\rho}}).$$

We can notice that in $5D$, the theory has one vector and one scalar gauge fields; one rank-2 and one rank-3 (that could be considered its dual that is rank-2) background fields. The next step is to make the dimensional reduction $5D \rightsquigarrow 4D$ à la Legendre of the $5D$ Lagrangian (2.14). Substituting (2.14) in

$$\mathcal{L}_4 = \mathcal{L}_5 - \frac{\partial \mathcal{L}_5}{\partial \partial_4 A_{\hat{\nu}}} \frac{\partial A_{\hat{\nu}}}{\partial x^4} - \frac{\partial \mathcal{L}_5}{\partial \partial_4 \varphi} \frac{\partial \varphi}{\partial x^4}, \tag{2.15}$$

[2] The symbol $\rightarrow$ means that the dimensional reduction is performed through the technique à la Scherk, while the symbol $\rightsquigarrow$ stands for the reduction à la Legendre.

and expliciting the index 4, we obtain the Lagrangian

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{4\mu}F^{4\mu} + \partial_\mu\phi F^{4\mu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\partial_4\varphi\partial^4\varphi \\ &\quad + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa p_\lambda + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\phi R_{\kappa\lambda} + \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\varphi S_{\kappa\lambda} + \phi\partial_\mu\varphi t^\mu,\end{aligned} \tag{2.16}$$

where $\mu = 0,1,2,3$, $\varepsilon^{\mu\nu\kappa\lambda 4} \equiv \varepsilon^{\mu\nu\kappa\lambda}$, and we redefine the gauge fields as

$$(A_{\hat{\mu}},\, \varphi) \rightsquigarrow (A_\mu,\, \phi,\, \varphi),$$

and the background fields as

$$(R_{\hat{\lambda}\hat{\rho}},\, T_{\hat{\kappa}\hat{\lambda}\hat{\rho}}) \rightsquigarrow (p_\lambda,\, R_{\lambda\rho},\, S_{\kappa\lambda},\, t^\mu). \tag{2.17}$$

Now, we can notice that the theory in $4D$ has a vector and two scalars gauge fields; two vectors (a vector was given by the dual of one rank-3 tensor) and two rank-2 tensor background fields.

The canonical reduction with respect to x^4 is:

$$\pi(A_\mu) = \frac{\partial\mathcal{L}_5}{\partial\partial_4 A_\mu} = -F^{4\mu} + \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}A_\nu R_{\kappa\lambda}, \tag{2.18}$$

$$\pi(\varphi) = \frac{\partial\mathcal{L}_5}{\partial\partial_4\varphi} = \partial^4\varphi - A_\mu t^\mu. \tag{2.19}$$

We define the auxiliary fields as

$$F_{4\mu} \equiv \eta_\mu, \qquad {F_4}^{\mu} = -F^{4\mu} \equiv \eta^\mu, \qquad \partial_4\varphi = -\partial^4\varphi \equiv G. \tag{2.20}$$

This will be necessary to close our future superalgebra off-shell. Replacing these definitions to the Lagrangian (2.16), one gets:

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\eta_\mu\eta^\mu - \eta^\mu\partial_\mu\phi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}GG \\ &\quad + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa p_\lambda + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\phi R_{\kappa\lambda} + \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\varphi S_{\kappa\lambda} + \phi\partial_\mu\varphi t^\mu.\end{aligned} \tag{2.21}$$

The $4D$ Lagrangian (2.21) will be invariant off-shell only if the constraints imposed by the equations of motion in $5D$, relative to the Lagrangian (2.14), were satisfied. The equation of motion for $A_{\hat{\nu}}$ is presented as follows:

$$\partial_{\hat{\mu}}F^{\hat{\mu}\hat{\nu}} + \frac{1}{2}\varepsilon^{\hat{\nu}\hat{\mu}\hat{\kappa}\hat{\lambda}\hat{\rho}}F_{\hat{\mu}\hat{\kappa}}R_{\hat{\lambda}\hat{\rho}} + \frac{1}{3!}\varepsilon^{\hat{\nu}\hat{\mu}\hat{\kappa}\hat{\lambda}\hat{\rho}}\partial_{\hat{\mu}}\varphi T_{\hat{\kappa}\hat{\lambda}\hat{\rho}} = 0. \tag{2.22}$$

Considering $\hat{\nu} = 4$, we have

$$\partial_\mu\eta^\mu - \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}R_{\kappa\lambda} - \partial_\mu\varphi t^\mu = 0, \tag{2.23}$$

and for $\hat{\nu} = \nu$,

$$\partial_4 \eta^\nu = \partial_\mu F^{\mu\nu} + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda} F_{\mu\kappa} p_\lambda + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda}\partial_\mu \varphi S_{\kappa\lambda} - \varepsilon^{\nu\mu\kappa\lambda}\eta_\mu R_{\kappa\lambda} + G t^\mu. \tag{2.24}$$

The equation of motion for φ is presented as below

$$\partial_{\hat{\mu}}\partial^{\hat{\mu}}\varphi - \frac{1}{3!}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}\partial_{\hat{\mu}} A_{\hat{\nu}} T_{\hat{\kappa}\hat{\lambda}\hat{\rho}} = 0. \tag{2.25}$$

Taking the component $\hat{\mu} = 4$, this equation looks as

$$\partial_4 G = \partial_\mu \partial^\mu \varphi - \frac{1}{4}\varepsilon^{\mu\kappa\lambda\rho} F_{\mu\nu} S_{\lambda\rho} - \eta_\mu t^\mu. \tag{2.26}$$

Notice that the constraint (2.23) can simplify the Lagrangian to

$$\begin{aligned} \mathcal{L}_4 \;\; = \;\; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\eta_\mu \eta^\mu + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}GG \\ & + \varepsilon^{\mu\nu\kappa\lambda} A_\mu \partial_\nu A_\kappa p_\lambda + \frac{1}{4}\varepsilon^{\mu\nu\kappa\lambda} F_{\mu\nu}\varphi S_{\kappa\lambda}, \end{aligned} \tag{2.27}$$

where ϕ works as a Lagrange multiplier. Notice that the auxiliary field η_μ must satisfy the equation (2.23). The equations (2.24) and (2.26) can be seen as central charge transformations. In this way, the supersymmetric transformations given explicitly in 4D will contain x^4 translations that are interpreted as central charge transformations. Then, we notice that this extra dimension works as a central charge for the superalgebra in $4D$. This results are possible because we used the technique of dimensional reduction à la Legendre. We can observe that the action of this Lagrangian is independent of the x^4-translations, although the fields depend on this coordinate. For that, the Lagrangian is required to obey all the constraint imposed by the equation of motion (2.22) in 5 dimensions, to be invariant under x^4 translations.

2.2 The chain of reductions $10D \to 5D \rightsquigarrow 4D$

Using the same idea as worked out for $N = 2$, we can obtain the bosonic sector of the off-shell $N = 4-$SUSY version of the model (2.1), but, to achieve this, we need to start off from a $D = 10$ Lagrangian. It can be done making the reduction $D = 10 \to D = 5$ through the ordinary technique and then reducing $D = 5 \rightsquigarrow D = 4$ through the Legendre transformation technique. As we see in [10], the Lagrangian treated in $10D$ can be written as:

$$\mathcal{L}_{10} = -\frac{1}{4}F_{\dot{\mu}\dot{\nu}}F^{\dot{\mu}\dot{\nu}} + \frac{1}{7!}\varepsilon^{\dot{\mu}\dot{\nu}\dot{\kappa}\dot{\lambda}\dot{\rho}\dot{\sigma}\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} A_{\dot{\mu}}\partial_{\dot{\nu}} A_{\dot{\kappa}} T_{\dot{\lambda}\dot{\rho}\dot{\sigma}\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}, \tag{2.28}$$

where $\dot{\mu} = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

It is easier in the calculations to consider the dual of the Lorentz violating term, although we will continue to show the results under the conventional form for the Chern-Simons

action. The dimensional reduction à la Scherk from $10 \to 5$ of the space-like coordinates x^5, x^6, x^7, x^8, x^9 is carried out by considering no dependence of the fields on these coordinates. This reduction takes the form as follows:

$$\begin{aligned}\mathcal{L}_5 &= -\frac{1}{4}F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}} + \frac{1}{2}\partial_{\hat{\mu}}\varphi^I\partial^{\hat{\mu}}\varphi^I + \frac{1}{2}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}A_{\hat{\mu}}\partial_{\hat{\nu}}A_{\hat{\kappa}}R_{\hat{\lambda}\hat{\rho}} \\ &\quad + \frac{1}{3!}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}A_{\hat{\mu}}\partial_{\hat{\nu}}\varphi^I T^I_{\hat{\kappa}\hat{\lambda}\hat{\rho}} + \frac{1}{4!}\varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}\varphi^I\partial_{\hat{\mu}}\varphi^J T^{IJ}_{\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}.\end{aligned} \tag{2.29}$$

where $\hat{\mu} = 0, 1, 2, 3, 4$, the internal index $I = 1, 2, 3, 4, 5$ and $\varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}56789} \equiv \varepsilon^{\hat{\mu}\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}$. The gauge fields were redefined as:

$$(A_{\dot{\mu}}) \to (A_{\hat{\mu}},\, \varphi^I),$$

and the background fields as

$$(T_{\dot{\lambda}\dot{\rho}\dot{\sigma}\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}) \to (R_{\hat{\lambda}\hat{\rho}},\, T^I_{\hat{\kappa}\hat{\lambda}\hat{\rho}},\, T^{IJ}_{\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}).$$

We can notice that the Lagrangian in $5D$ has a vector and five scalar gauge fields; one rank-2, five rank-3 (its dual is rank-2) and ten rank-4 (its dual is a vector) background fields.

Now, we proceed performing the dimensional reduction of the space-like x^4-coordinate by using the Legendre transformation technique. This will bring about the auxiliary fields that make the superalgebra closed off-shell. In this technique, the $D = 4$ Lagrangian is obtained through

$$\mathcal{L}_4 = \mathcal{L}_5 - \frac{\partial\mathcal{L}_5}{\partial\partial_4 A_{\hat{\nu}}}\frac{\partial A_{\hat{\nu}}}{\partial x^4} - \frac{\partial\mathcal{L}_5}{\partial\partial_4\varphi^I}\frac{\partial\varphi^I}{\partial x^4}. \tag{2.30}$$

Applying the (2.30) for the Lagrangian (2.29), we obtain:

$$\mathcal{L}_4 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{4\mu}F^{4\mu} + \partial_\mu\phi F^{4\mu} + \frac{1}{2}\partial_\mu\varphi^I\partial^\mu\varphi^I - \frac{1}{2}\partial_4\varphi^I\partial^4\varphi^I \tag{2.31}$$

$$+\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa p_\lambda + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\phi R_{\kappa\lambda} \tag{2.32}$$

$$+\frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\varphi^I S^I_{\kappa\lambda} + \phi\partial_\mu\varphi^I t^{I\mu} + \varphi^I\partial_\mu\varphi^J t^{IJ\mu},$$

where $\varepsilon^{\mu\nu\kappa\lambda} \equiv \varepsilon^{\mu\nu\kappa\lambda 4}$, and we redefined the gauge fields as

$$(A_{\hat{\mu}},\, \varphi^I) \rightsquigarrow (A_\mu,\, \phi, \varphi^I),$$

and the background fields as

$$(R_{\hat{\lambda}\hat{\rho}},\, T^I_{\hat{\kappa}\hat{\lambda}\hat{\rho}},\, T^{IJ}_{\hat{\nu}\hat{\kappa}\hat{\lambda}\hat{\rho}}) \rightsquigarrow (p_\lambda,\, R_{\lambda\rho},\, S^I_{\kappa\lambda},\, t^{I\mu},\, t^{IJ\mu},\, t^{IJ}).$$

We notice that the theory in $4D$ has one vector and six scalar gauge fields; sixteen vector (given by one vector and fifteen duals of rank-3 tensors), ten scalars (given by ten duals of rank-4 tensors) and six rank-2 tensor background fields.

The canonical momenta for the Lagrangian (2.29) are given as follows:

$$\begin{aligned}\pi(A_\mu) &= \frac{\partial \mathcal{L}_5}{\partial \partial_4 A_\mu} = -F^{4\mu} + \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}A_\nu R_{\kappa\lambda}, \\ \pi(\varphi) &= \frac{\partial \mathcal{L}_5}{\partial \partial_4 \varphi^I} = \partial^4\varphi^I - A_\mu t^{I\mu} + \varphi^J t^{IJ}.\end{aligned}$$

We define the new auxiliary fields as

$$F_{4\mu} \equiv \eta_\mu, \quad F_4{}^\mu = -F^{4\mu} \equiv \eta^\mu, \quad \partial_4\varphi^I = -\partial^4\varphi^I \equiv G^I.$$

This will be necessary to close the superalgebra off-shell. By bringing these definitions into the Lagrangian (2.31), we end up with

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\eta_\mu\eta^\mu - \eta^\mu\partial_\mu\phi + \frac{1}{2}\partial_\mu\varphi^I\partial^\mu\varphi^I + \frac{1}{2}G^IG^I + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa p_\lambda \\ &+\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\phi R_{\kappa\lambda} + \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu\varphi^I S^I_{\kappa\lambda} + \phi\partial_\mu\varphi^I t^{I\mu} + \varphi^I\partial_\mu\varphi^J t^{IJ\mu}.\end{aligned} \tag{2.33}$$

The equation of motion for the Lagrangian (2.29) in terms of $A_{\hat\nu}$ is given by:

$$\partial_{\hat\mu}F^{\hat\mu\hat\nu} + \varepsilon^{\hat\nu\hat\mu\hat\kappa\hat\lambda\hat\rho}F_{\hat\mu\hat\kappa}R_{\hat\lambda\hat\rho} + \frac{1}{3!}\varepsilon^{\hat\nu\hat\mu\hat\kappa\hat\lambda\hat\rho}\partial_{\hat\mu}\varphi^I T^I_{\hat\kappa\hat\lambda\hat\rho} = 0. \tag{2.34}$$

Considering $\hat\nu = 4$, we have that

$$\partial_\mu\eta^\mu - \frac{1}{2}\varepsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}R_{\kappa\lambda} - \partial_\mu\varphi^I t^{I\mu} = 0, \tag{2.35}$$

and considering $\hat\nu = \nu$, we have that

$$\partial_4\eta^\nu = \partial_\mu F^{\mu\nu} + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda}F_{\mu\kappa}p_\lambda + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda}\partial_\mu\varphi^I S^I_{\kappa\lambda} - \varepsilon^{\nu\mu\kappa\lambda}\eta_\mu R_{\kappa\lambda} + G^I t^{I\mu}. \tag{2.36}$$

The equation of motion for φ is presented as follows

$$\partial_{\hat\mu}\partial^{\hat\mu}\varphi^I - \frac{1}{3!}\varepsilon^{\hat\mu\hat\nu\hat\kappa\hat\lambda\hat\rho}\partial_{\hat\mu}A_{\hat\nu}T^I_{\hat\kappa\hat\lambda\hat\rho} - \frac{2}{4!}\varepsilon^{\hat\mu\hat\nu\hat\kappa\hat\lambda\hat\rho}\partial_{\hat\mu}\varphi^J T^{IJ}_{\hat\nu\hat\kappa\hat\lambda\hat\rho} = 0. \tag{2.37}$$

Taking the component $\hat\mu = 4$, we arrive at

$$\partial_4 G^I = \partial_\mu\partial^\mu\varphi^I - \frac{1}{4}\varepsilon^{\mu\kappa\lambda\rho}F_{\mu\nu}S^I_{\lambda\rho} - \eta_\mu t^{I\mu} - 2\partial_\mu\varphi^J t^{IJ\mu} - 2G^J t^{IJ}. \tag{2.38}$$

Notice that the constraint (2.23) can directly simplify the Lagrangian to

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\eta_\mu\eta^\mu + \frac{1}{2}\partial_\mu\varphi^I\partial^\mu\varphi^I + \frac{1}{2}G^IG^I \\ &+\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa p_\lambda + \frac{1}{4}\varepsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}\varphi^I S^I_{\kappa\lambda} + \varphi^I\partial_\mu\varphi^J t^{IJ\mu}.\end{aligned} \tag{2.39}$$

As in the 6 to 4 reduction the field ϕ works as a Lagrange multiplier. The auxiliary field η_μ must satisfy the equation (2.35) and the x^4-translations (2.36,2.38) correspond the central charge transformations in the $N = 4$ superalgebra for $D = 4$.

3 The off-shell $N = 2$-SUSY version of the Abelian gauge model with Lorentz-breaking term

The on-shell version of the $N = 2$ supersymmetric extension of the Lorentz breaking term can be found in [10], where use has been made of the dimensional reduction (à la Scherk). In the present work, we are interested in the attainment of the off-shell $N = 2$ SUSY version of the Abelian gauge model with Lorentz-breaking term. In this way, we consider that the bosonic sector for $N = 1$ in $6D$ is the same of the bosonic sector for $N = 2$ in $4D$. In order to build up the supersymmetrization off-shell, it is necessary to reduce one of the coordinates using the reduction à la Legendre. It permits the appearance of auxiliary fields that make the algebra closed off-shell.

In the previous section, we have made the dimensional reduction $6 \to 5$ à la Scherk and $5 \rightsquigarrow 4$ à la Legendre and obtained the bosonic Lagrangian (2.27) in $4D$. We can consider the vector field of the theory as a gradient of a scalar ($p_\mu = \partial_\mu s$), then the Lagrangian is given as follows:

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\eta_\mu\eta^\mu + \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}G_1G_1 \\ &\quad +\varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa\partial_\lambda s_1 + \frac{1}{4}\varepsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}\varphi_1 S_{\kappa\lambda}.\end{aligned} \tag{3.1}$$

Notice that we redefined the real fields φ, s and G. It is necessary in order to accommodate these bosonic component fields in the chiral superfields. For that, we have to define the complex fields:

$$\begin{aligned}\varphi &= \varphi_1 + i\varphi_2 \\ s &= s_2 + is_1, \\ G &= G_1 + iG_2.\end{aligned}$$

Observe that we introduced more one gauge, one background and one auxiliary fields in the model to build up complex scalar fields. Once we has the bosonic sector of the $N = 2$-SUSY theory, we proceed the supersymmetrization using the superfield formulation in a $N = 1$ superspace with supercoordinates $(x^\mu, \theta^a, \bar{\theta}_{\dot{a}})$. The conventions used are the same as given in ref. [10].

We define a vector superfields V in the Wess-Zumino gauge containing the gauge field A_μ as:

$$V = \theta\sigma^\mu\bar{\theta}A_\mu + \theta^2\bar{\theta}\bar{\lambda} + \bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}^2 D \tag{3.2}$$

which fulfill the reality constraint $V = V^\dagger$. The Abelian field-strength superfield is given by:

$$W_a = -\frac{1}{4}\bar{D}^2 D_a V_{WZ}, \qquad \bar{W}_{\dot{a}} = -\frac{1}{4}D^2\bar{D}_{\dot{a}}V_{WZ}, \tag{3.3}$$

having the chirality condition: $\bar{D}W = D\bar{W} = 0$ and $DW = \bar{D}\bar{W}$.

The vector superfield that has the auxiliary field η^μ must not be gauge invariant, then, it is necessary to define this in the complete form:

$$U = u + \theta\alpha + \bar{\theta}\bar{\alpha} + \theta^2 M + \bar{\theta}^2 M^* + \theta\sigma^\mu\bar{\theta}\eta_\mu + \theta^2\bar{\theta}\bar{\beta} + \bar{\theta}^2\theta\beta + \theta^2\bar{\theta}^2 E, \tag{3.4}$$

obeying the reality constraint, $U = U^\dagger$.

The scalar superfields that accommodate the gauge field φ, φ^* and the auxiliary fields G, G^* are:

$$\Phi = \varphi + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi - \frac{1}{4}\theta^2\bar{\theta}^2\Box\varphi + \sqrt{2}\theta\psi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2 G, \tag{3.5}$$

$$\bar{\Phi} = \varphi^* - i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi^* - \frac{1}{4}\theta^2\bar{\theta}^2\Box\varphi^* + \sqrt{2}\bar{\theta}\bar{\psi} + \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\psi} + \bar{\theta}^2 G^*. \tag{3.6}$$

The scalar superfields that accommodate the background fields s, s^* and their superpartners are written, respectively as

$$S = s + i\theta\sigma^\mu\bar{\theta}\partial_\mu s - \frac{1}{4}\theta^2\bar{\theta}^2\Box s + \sqrt{2}\theta\xi - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\xi\sigma^\mu\bar{\theta} + \theta^2 h, \tag{3.7}$$

$$\bar{S} = s^* - i\theta\sigma^\mu\bar{\theta}\partial_\mu s^* - \frac{1}{4}\theta^2\bar{\theta}^2\Box s^* + \sqrt{2}\bar{\theta}\bar{\xi} + \frac{i}{\sqrt{2}}\bar{\theta}^2\theta\sigma^\mu\partial_\mu\bar{\xi} + \bar{\theta}^2 h^*, \tag{3.8}$$

which satisfy the chiral condition: $\bar{D}\Phi = D\bar{\Phi} = \bar{D}S = D\bar{S} = 0$. Observe that we introduced more one gauge, one auxiliary and one background fields in the model to build up complex scalar fields. It was necessary in order to accommodate the fields in the chiral superfields.

The spinor superfields that contain $S_{\mu\nu}$, their dual fields and their superpartners are written as

$$\begin{aligned}\Sigma_a &= \tau_a + \theta^b(\varepsilon_{ba}\rho + \sigma^{\mu\nu}_{ba}S_{\mu\nu}) + \theta^2 F_a + i\theta\sigma^\mu\bar{\theta}\partial_\mu\tau_a \\ &\quad + i\theta\sigma^\mu\bar{\theta}\theta^b\partial_\mu(\varepsilon_{ba}\rho + \sigma^{\mu\nu}_{ba}S_{\mu\nu}) - \frac{1}{4}\theta^2\bar{\theta}^2\Box\tau_a,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\bar{\Sigma}_{\dot{a}} &= \bar{\tau}_{\dot{a}} + \bar{\theta}_{\dot{b}}(-\varepsilon^{\dot{b}}{}_{\dot{a}}\rho^* - \bar{\sigma}^{\mu\nu\dot{b}}{}_{\dot{a}}S^*_{\mu\nu}) + \bar{\theta}^2\bar{F}_{\dot{a}} - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\tau}_{\dot{a}} \\ &\quad - i\theta\sigma^\mu\bar{\theta}\theta_{\dot{b}}\partial_\mu(-\varepsilon^{\dot{b}}{}_{\dot{a}}\rho^* - \bar{\sigma}^{\mu\nu\dot{b}}{}_{\dot{a}}S^*_{\mu\nu}) - \frac{1}{4}\theta^2\bar{\theta}^2\Box\bar{\tau}_{\dot{a}},\end{aligned} \tag{3.10}$$

that are also chiral $\bar{D}_{\dot{b}}\Sigma_a = D_b\bar{\Sigma}_{\dot{a}} = 0$.

Now, we are interested in building up the off-shell $N = 2$ supersymmetric version of the Lagrangian (2.1) using the $N = 1$-superfield formalism. As already written down in the previous section, the bosonic sector for this Lagrangian is given by (2.27). In this way, the next step is to look for a supersymmetric model in term of superfields that contain this

bosonic sector and its correspondent fermionic sector. First, it is useful to quote the mass dimensions of the superfields given previously:

$$\begin{aligned}
[V] &= 0, \quad [W_a] = [\bar{W}_{\dot{a}}] = +\frac{3}{2}, \quad [\Phi] = [\bar{\Phi}] = +1, \\
[U] &= [\bar{U}] = +1, \quad [S] = [\bar{S}] = 0, \quad [\Sigma_a] = [\bar{\Sigma}_{\dot{a}}] = +1.
\end{aligned}$$

Based on the dimensionalities, and by analyzing the bosonic Lagrangian (2.27), we propose the following supersymmetric action $\mathcal{S}_{br}$:

$$\begin{aligned}
\mathcal{S}_{br} &= \int d^4x d^2\theta d^2\bar{\theta}[\frac{1}{4}W^\alpha W_\alpha \delta(\bar{\theta}^2) + \frac{1}{4}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\delta(\theta^2) + \frac{1}{2}\bar{\Phi}\Phi - UU \\
&+\frac{1}{2}W^a(D_aV)S + \frac{1}{2}\bar{W}_{\dot{a}}(\bar{D}^{\dot{a}}V)\bar{S} \\
&+\frac{i}{4}\delta(\bar{\theta})W^a(\Phi+\bar{\Phi})\Sigma_a - \frac{i}{4}\delta(\theta)\bar{W}_{\dot{a}}(\Phi+\bar{\Phi})\bar{\Sigma}^{\dot{a}}].
\end{aligned} \tag{3.11}$$

This Lagrangian in its component-field reads as below:

$$\begin{aligned}
\mathcal{L}_{br} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 + D^{*2} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi^* - \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{1}{2}GG^* \\
&-\frac{1}{2}\eta^\mu\eta_\mu + \alpha\beta + \bar{\alpha}\bar{\beta} - MM^* - 2uE \\
&+\frac{i}{4}\partial_\mu(s-s^*)\varepsilon^{\mu\kappa\lambda\nu}F_{\kappa\lambda}A_\nu - \frac{1}{4}(s+s^*)F_{\mu\nu}F^{\mu\nu} + 2D^2(s+s^*) \\
&-is\lambda\sigma^\mu\partial_\mu\bar{\lambda} - is^*\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda - \frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}F_{\mu\nu}\xi + \frac{1}{\sqrt{2}}\bar{\lambda}\bar{\sigma}^{\mu\nu}F_{\mu\nu}\bar{\xi} \\
&+\frac{1}{2}\lambda\lambda h + \frac{1}{2}\bar{\lambda}\bar{\lambda}h^* - \sqrt{2}\lambda\xi D - \sqrt{2}\bar{\lambda}\bar{\xi}D \\
&\frac{1}{16}\varepsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}(\varphi+\varphi^*)(R_{\kappa\lambda}+R^*_{\kappa\lambda}) + \frac{i}{8}F^{\mu\nu}(R_{\mu\nu}-R^*_{\mu\nu})(\varphi+\varphi^*) \\
&-\frac{i\sqrt{2}}{8}\tau\sigma^{\mu\nu}\psi F_{\mu\nu} - \frac{i\sqrt{2}}{8}\bar{\tau}\bar{\sigma}^{\mu\nu}\bar{\psi}F_{\mu\nu} + \frac{1}{4}\tau\sigma^\mu\partial_\mu\bar{\lambda}(\varphi+\varphi^*) \\
&-\frac{1}{4}\bar{\tau}\bar{\sigma}^\mu\partial_\mu\lambda(\varphi+\varphi^*) + \frac{i\sqrt{2}}{4}\psi\sigma^{\mu\nu}B_{\mu\nu}\lambda + \frac{i\sqrt{2}}{4}\bar{\psi}\bar{\sigma}^{\mu\nu}B^*_{\mu\nu}\bar{\lambda} \\
&-\frac{i}{2}D(\varphi+\varphi^*)\rho + \frac{i}{2}D^*(\varphi+\varphi^*)\rho^* \\
&+\frac{i\sqrt{2}}{8}\lambda\psi\rho - \frac{i\sqrt{2}}{8}\bar{\lambda}\bar{\psi}\rho^* - \frac{i\sqrt{2}}{4}D\psi\tau + \frac{i\sqrt{2}}{4}D^*\bar{\psi}\bar{\tau} \\
&+\frac{i}{4}f\lambda\tau - \frac{i}{4}f^*\bar{\lambda}\bar{\tau} + \frac{i}{4}(\varphi+\varphi^*)\lambda F - \frac{i}{4}(\varphi+\varphi^*)\bar{\lambda}\bar{F}.
\end{aligned} \tag{3.12}$$

This Lagrangian is invariant under the $N=2$ superalgebra in this $4D$. When the superalgebra in $5D$ has the index 4 explicited we obtain the superalgebra in $4D$ that acquire central

charge transformations that correspond the x^4 transformations as we obtained in (2.24, 2.26). This is given as follows:

$$\begin{aligned}
\delta_Z \eta^\nu &= \partial_\mu F^{\mu\nu} + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda}F_{\mu\kappa}p_\lambda - \varepsilon^{\nu\mu\kappa\lambda}\eta_\mu R_{\kappa\lambda} \\
&\quad + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda}\partial_\mu\varphi S_{\kappa\lambda} - Gt^\mu + fermionic\ SUSY\ partners,
\end{aligned} \tag{3.13}$$

$$\delta_Z G = \partial_\mu \partial^\mu \varphi - \frac{1}{4}\varepsilon^{\mu\kappa\lambda\rho}F_{\mu\nu}S_{\lambda\rho} - \eta_\mu t^\mu + fermionic\ SUSY\ partners. \tag{3.14}$$

These central charge transformations above include only the bosonic sector, because we do not start off with the fermions in $6D$. Once we have the reduced fermionic term (in $4D$), we are able to restore the fermions of the original theory in $6D$. With that, our central charge transformations in $4D$ will naturally display the fermionic fields along with the bosonic degrees of freedom. This shall be presented in a forthcoming work.

The fifth dimension is interpreted as the central charge and it is consequence of the necessity of the obedience of the equation of motion in $5D$.

We can notice that this off-shell Lagrangian is simpler than the on-shell one obtained in the in ref. [10]. This is so by virtue of the constraint (2.23) imposed by the equation of motion in 5 dimensions. These results are possible due to the utility of the Legendre transformation technique to make the dimensional reduction of one of the space-like coordinates.

In the Lagrangian (3.12), we can notice the Maxwell term and the term proposed by Jackiw in [1] and also the $N = 1$ supersymmetric generalization. We also see that this Lagrangian has the bosonic sector (2.27), the fermionic sector and the auxiliary fields presented in the superfields.

4 The off-shell $N = 4$-SUSY version of the Abelian gauge model with Lorentz-breaking term

In order to build up the off-shell $N = 4$ supersymmetric version of the Abelian gauge model with Lorentz violating term, we proceed similar to the last section, but now, we start with the bosonic Lagrangian (2.39) obtained through the reduction $10 \to 5$ and $5 \rightsquigarrow 4$. As in the last section, we consider the vector fields as gradient of scalars ($p_\mu = \partial_\mu s,\ t^{IJ\mu} = \partial^\mu u^{IJ}$), then the bosonic Lagrangian is given as follows:

$$\begin{aligned}
\mathcal{L}_4 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\eta_\mu\eta^\mu + \frac{1}{2}\partial_\mu\varphi_1^I\partial^\mu\varphi_1^I + \frac{1}{2}G_1^I G_1^I \\
&\quad + \varepsilon^{\mu\nu\kappa\lambda}A_\mu\partial_\nu A_\kappa\partial_\lambda s_2 + \frac{1}{4}\varepsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}\varphi_1^I S_{\kappa\lambda}^I + \varphi_1^I\partial_\mu\varphi_1^J\partial^\mu u_1^{IJ}.
\end{aligned} \tag{4.1}$$

As it was done in the previous section, we redefined the real fields φ^I, s and G^I in order to accommodate these bosonic component fields in the chiral superfields. We define the

complex fields:

$$\begin{aligned}\varphi^I &= \varphi_1^I + i\varphi_2^I,\\ s &= s_2 + is_1,\\ G^I &= G_1^I + iG_2^I,\\ u^{IJ} &= u_1^{IJ} + iu_2^{IJ}\end{aligned}$$

We proceed the supersymmetrization building up the $N = 1-$superfield extension of the Lagrangian (4.1). The superfields that accommodates the gauge field A_μ, the background scalar fields s, s^* and the auxiliary vector field η^μ with theirs superpartners is the same of the defined in the last section, given by (3.2, 3.7, 3.8, 3.4) respectively. The scalar superfields that accommodate the five real scalar gauge fields φ_1^I, and introducing more five real fields φ_2^I that does not appears in the Lagrangian (2.39), are given, respectively as

$$\Phi^I = \varphi^I + i\theta\sigma^\mu\bar\theta\partial_\mu\varphi^I - \frac{1}{4}\theta^2\bar\theta^2\Box\varphi^I + \sqrt{2}\theta\psi^I - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi^I\sigma^\mu\bar\theta + \theta^2 G^I, \tag{4.2}$$

$$\bar\Phi^I = \varphi^{*I} - i\theta\sigma^\mu\bar\theta\partial_\mu\varphi^{*I} - \frac{1}{4}\theta^2\bar\theta^2\Box\varphi^{*I} + \sqrt{2}\bar\theta\bar\psi^I + \frac{i}{\sqrt{2}}\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\psi^I + \bar\theta^2 G^{*I}, \tag{4.3}$$

which satisfy the chiral condition: $\bar D\Phi^I = D\bar\Phi^I = 0$. The chiral superfields that accommodate u^{IJ} and u^{*IJ} are

$$R^{IJ} = u^{IJ} + i\theta\sigma^\mu\bar\theta\partial_\mu u^{IJ} - \frac{1}{4}\theta^2\bar\theta^2\Box u^{IJ} + \sqrt{2}\theta\zeta^{IJ} - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\zeta^{IJ}\sigma^\mu\bar\theta + \theta^2 g^I, \tag{4.4}$$

$$\bar R^{IJ} = u^{*IJ} - i\theta\sigma^\mu\bar\theta\partial_\mu u^{*IJ} - \frac{1}{4}\theta^2\bar\theta^2\Box u^{*IJ} + \sqrt{2}\bar\theta\bar\zeta^{IJ} + \frac{i}{\sqrt{2}}\bar\theta^2\theta\sigma^\mu\partial_\mu\bar\zeta^{IJ} + \bar\theta^2 g^{*I}, \tag{4.5}$$

The spinor superfields that contain $S^I_{\mu\nu}$, their dual fields and their superpartners are written as

$$\begin{aligned}\Sigma_a^I &= \tau_a^I + \theta^b(\varepsilon_{ba}\rho^I + \sigma^{\mu\nu}_{ba}S^I_{\mu\nu}) + \theta^2 F_a^I + i\theta\sigma^\mu\bar\theta\partial_\mu\tau_a^I\\ &\quad + i\theta\sigma^\mu\bar\theta\theta^b\partial_\mu(\varepsilon_{ba}\rho^I + \sigma^{\mu\nu}_{ba}S^I_{\mu\nu}) - \frac{1}{4}\theta^2\bar\theta^2\Box\tau_a^I,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\bar\Sigma_{\dot a}^I &= \bar\tau_{\dot a}^I + \bar\theta_{\dot b}(-\varepsilon^{\dot b}{}_{\dot a}\rho^{*I} - \bar\sigma^{\mu\nu\dot b}{}_{\dot a}S^{*I}_{\mu\nu}) + \bar\theta^2\bar F_{\dot a}^I - i\theta\sigma^\mu\bar\theta\partial_\mu\bar\tau_{\dot a}^I\\ &\quad - i\theta\sigma^\mu\bar\theta\theta_{\dot b}\partial_\mu(-\varepsilon^{\dot b}{}_{\dot a}\rho^{*I} - \bar\sigma^{\mu\nu\dot b}{}_{\dot a}S^{*I}_{\mu\nu}) - \frac{1}{4}\theta^2\bar\theta^2\Box\bar\tau_{\dot a}^I,\end{aligned} \tag{4.7}$$

that are also chiral $\bar D_{\dot b}\Sigma_a^I = D_b\bar\Sigma_{\dot a}^I = 0$.

Now, we are interested in building up the off-shell $N = 4$ supersymmetric version of the Lagrangian (2.1). The bosonic sector for this Lagrangian is given by (2.39). Now, we look for a supersymmetric model in terms of superfields. The dimensions of the new superfields are given as:

$$[\Phi^I] = [\bar\Phi^I] = +1, \qquad [\Sigma_a^I] = [\bar\Sigma_{\dot a}^I] = +1, \qquad [R^{IJ}] = [\bar R^{IJ}] = 0.$$

Based on the dimensionalities, and by analyzing the bosonic Lagrangian (2.39), we propose the following supersymmetric action, S_{br}:

$$
\begin{aligned}
\mathcal{S}_{br} &= \int d^4x d^2\theta d^2\bar{\theta}[\frac{1}{4}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\delta(\theta^2) + \frac{1}{4}W^{\alpha}W_{\alpha}\delta(\bar{\theta}^2) + \frac{1}{2}\bar{\Phi}^I\Phi^I \\
&\quad -UU + \frac{1}{2}W^a(D_aV)S + \frac{1}{2}\bar{W}_{\dot{a}}(\bar{D}^{\dot{a}}V)\bar{S} \qquad (4.8)\\
&\quad +\frac{i}{4}\delta(\bar{\theta})W^a(\Phi^I + \bar{\Phi}^I)\Sigma_a^I - \frac{i}{4}\delta(\theta)\bar{W}_{\dot{a}}(\Phi^I + \bar{\Phi}^I)\bar{\Sigma}^{I\dot{a}}] \\
&\quad +\frac{1}{2}\Phi^I\bar{\Phi}^J(R^{IJ} - \bar{R}^{IJ}). \qquad (4.9)
\end{aligned}
$$

This Lagrangian in its component-field version reads as below:

$$
\begin{aligned}
\mathcal{L}_{br} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda} + D^2 + D^{*2} + \frac{1}{2}\partial_{\mu}\varphi^I\partial^{\mu}\varphi^{*I} - \frac{i}{2}\psi^I\sigma^{\mu}\partial_{\mu}\bar{\psi}^I + \frac{1}{2}G^IG^{*I} \\
&\quad -\frac{1}{2}\eta^{\mu}\eta_{\mu} + \alpha\beta + \bar{\alpha}\bar{\beta} - MM^* - 2uE \\
&\quad +\frac{i}{4}\partial_{\mu}(s-s^*)\varepsilon^{\mu\kappa\lambda\nu}F_{\kappa\lambda}A_{\nu} - \frac{1}{4}(s+s^*)F_{\mu\nu}F^{\mu\nu} + 2D^2(s+s^*) \\
&\quad -is\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda} - is^*\bar{\lambda}\bar{\sigma}^{\mu}\partial_{\mu}\lambda - \frac{1}{\sqrt{2}}\lambda\sigma^{\mu\nu}F_{\mu\nu}\xi + \frac{1}{\sqrt{2}}\bar{\lambda}\bar{\sigma}^{\mu\nu}F_{\mu\nu}\bar{\xi} \\
&\quad +\frac{1}{2}\lambda\lambda h + \frac{1}{2}\bar{\lambda}\bar{\lambda}h^* - \sqrt{2}\lambda\xi D - \sqrt{2}\bar{\lambda}\bar{\xi}D \\
&\quad \frac{1}{16}\varepsilon^{\mu\nu\kappa\lambda}F_{\mu\nu}(\varphi^I + \varphi^{*I})(S^I_{\kappa\lambda} + S^{*I}_{\kappa\lambda}) + \frac{i}{8}F^{\mu\nu}(S^I_{\mu\nu} - S^{*I}_{\mu\nu})(\varphi^I + \varphi^{*I}) \\
&\quad -\frac{i\sqrt{2}}{8}\tau^I\sigma^{\mu\nu}\psi^IF_{\mu\nu} - \frac{i\sqrt{2}}{8}\bar{\tau}^I\bar{\sigma}^{\mu\nu}\bar{\psi}^IF_{\mu\nu} + \frac{1}{4}\tau^I\sigma^{\mu}\partial_{\mu}\bar{\lambda}(\varphi^I + \varphi^{*I}) \qquad (4.10)\\
&\quad -\frac{1}{4}\bar{\tau}^I\bar{\sigma}^{\mu}\partial_{\mu}\lambda(\varphi^I + \varphi^{*I}) + \frac{i\sqrt{2}}{4}\psi^I\sigma^{\mu\nu}S^I_{\mu\nu}\lambda + \frac{i\sqrt{2}}{4}\bar{\psi}^I\bar{\sigma}^{\mu\nu}S^{I*}_{\mu\nu}\bar{\lambda}^I \\
&\quad -\frac{i}{2}D(\varphi^I + \varphi^{*I})\rho^I + \frac{i}{2}D^*(\varphi^I + \varphi^{*I})\rho^{*I} \\
&\quad +\frac{i\sqrt{2}}{8}\lambda\psi^I\rho^I - \frac{i\sqrt{2}}{8}\bar{\lambda}\bar{\psi}^I\rho^{*I} - \frac{i\sqrt{2}}{4}D\psi^I\tau^I + \frac{i\sqrt{2}}{4}D^*\bar{\psi}^I\bar{\tau}^I \\
&\quad +\frac{i}{4}f^I\lambda\tau^I - \frac{i}{4}f^{*I}\bar{\lambda}\bar{\tau}^I + \frac{i}{4}(\varphi^I + \varphi^{*I})\lambda F^I - \frac{i}{4}(\varphi^I + \varphi^{*I})\bar{\lambda}\bar{F}^I \\
&\quad +\frac{1}{4}\varphi^I\partial_{\mu}\varphi^{*J}\partial^{\mu}(u^{IJ} + u^{*IJ}) - \frac{1}{4}\varphi^{*J}\partial_{\mu}\varphi^I\partial^{\mu}(u^{IJ} + u^{*IJ}) \\
&\quad +\frac{1}{2}\partial^{\mu}\varphi^I\partial_{\mu}\varphi^{*J}(u^{IJ} - u^{*IJ}) - \frac{1}{4}\varphi^I\varphi^{*J}\Box(u^{IJ} - u^{*IJ}) - \frac{i}{2}\psi^I\sigma^{\mu}\partial_{\mu}\bar{\psi}^J(u^{IJ} - u^{*IJ}) \\
&\quad +\frac{1}{2}G^IG^{*J}(u^{IJ} - u^{*IJ}) + \frac{i}{2}\psi^I\sigma^{\mu}\bar{\psi}^J\partial_{\mu}u^{*IJ} \\
&\quad -\frac{i}{2}\varphi^I\zeta^{IJ}\sigma^{\mu}\partial_{\mu}\bar{\psi}^J + \frac{i}{2}\varphi^{*J}\psi^I\sigma^{\mu}\partial_{\mu}\bar{\zeta}^{IJ} + \frac{i}{2}\psi^I\sigma^{\mu}\bar{\zeta}^{IJ}\partial_{\mu}\varphi^{*J}
\end{aligned}
$$

$$+\frac{1}{2}\varphi^I G^{*J} g^{IJ} - \frac{1}{2}G^I \varphi^{*J} g^{*IJ} - \frac{1}{2}G^{*J}\psi^I\zeta^{IJ} + \frac{1}{2}G^I\bar{\psi}^J\bar{\zeta}^{IJ}.$$

This $N = 4$ Lagrangian contains the $N = 1$ and $N = 2$ terms. The x^4 transformations given by (2.36,2.38) works as central charge transformations in the superalgebra. These are given by

$$\begin{aligned} \delta_Z\eta^\nu &= \partial_\mu F^{\mu\nu} + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda}F_{\mu\kappa}\partial_\lambda s + \frac{1}{2}\varepsilon^{\nu\mu\kappa\lambda}\partial_\mu\varphi^I S^I_{\kappa\lambda} \\ &\quad -\varepsilon^{\nu\mu\kappa\lambda}\eta_\mu R_{\kappa\lambda} + G^I\partial^\mu u^I + fermionic\ SUSY\ partners, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \delta_Z G^I &= \partial_\mu\partial^\mu\varphi^I - \frac{1}{4}\varepsilon^{\mu\kappa\lambda\rho}F_{\mu\nu}S^I_{\lambda\rho} - \eta_\mu t^{I\mu} \\ &\quad -2\partial_\mu\varphi^J\partial^\mu u^{IJ} - 2G^J t^{IJ} + fermionic\ SUSY\ partners. \end{aligned} \tag{4.12}$$

As already mentioned in the previous section, the fermionic terms of the central charge transformations shall appear in a forthcoming paper.

5 Concluding Remarks and Comments

In this work, we carried out the $N = 2$ and $N = 4$ supersymmetric generalizations realized off-shell for the Abelian gauge model with a Chern-Simons Lorentz violating term starting with the bosonic sector of a $N = 1$ version of this theory on $D = 6$ and $D = 4$ respectively, both realized on-shell. Then, the dimensional reduction to $D = 4$ yielded the bosonic sectors for the $N = 2$ and $N = 4$-SUSY versions for this model. Once we had the bosonic sector, we could make the supersymmetric extension using the superfields formalism in $N = 1$ superspace. The results obtained could be realized off-shell because we used the Legendre transformation technique to reduce one of the space-like coordinates. In so doing, the Lagrangian acquired auxiliary fields that make the algebra to close off-shell. Other important result of this technique is the appearance of central charge transformation that is consequence of translation with respect to the coordinates reduced with this technique. It is interesting because in the real physical situation the fields may carry a dependence on the extra space dimensions while the action does not need to show such a dependence. In a forthcoming work, we shall be discussing more deeply issues related to the central charges and their relation to topological configurations that may be found in the $N = 2$ and $N = 4$ extensions of the Lorentz-violating model.

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